

# Universality of massive scalar field late-time tails in black-hole spacetimes

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The late-time tails of a massive scalar field in the spacetime of black holes are studied numerically. Previous analytical results for a Schwarzschild black hole are confirmed: The late-time behavior of the field as recorded by a static observer is given by  $\psi(t) \sim t^{-5/6} \sin[\omega(t) \times t]$ , where  $\omega(t)$  depends weakly on time. This result is carried over to the case of a Kerr black hole. In particular, it is found that the power-law index of  $-5/6$  depends on neither the multipole mode  $\ell$  nor on the spin rate of the black hole  $a/M$ . In all black hole spacetimes, massive scalar fields have the same late-time behavior irrespective of their initial data (i.e., angular distribution). Their late-time behavior is universal.

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## I. INTRODUCTION

The late-time behavior of massless fields in black hole spacetime has been studied in detail for both linearized [1,2] and fully nonlinear (spherical) evolutions [3,4]. In contrast, the late-time behavior of massive fields has been studied in much less detail. In this paper we study the late-time behavior of a scalar field numerically, in the spacetimes of Schwarzschild and Kerr black holes.

The first to consider the problem of a massive scalar field in the spacetime of a Schwarzschild black hole were Novikov and Starobinski, who studied the problem in the frequency domain, and found that there are poles in the complex plane closer to the real axis than in the massless case. They thus inferred that the decay rate would be slower in the massive case than in the massless case [5]. That problem was later studied numerically by Burko [6] both for a linearized massive scalar field, and for a self-gravitating, spherical massive field. In both cases it was found in Ref. [6] that the late-time behavior was given by an oscillating field, whose envelope decayed according to an inverse power law, and whose frequency was determined by the Compton wavelength of the field, i.e., by its mass term. In Ref. [6] it was also reported that the decay rate of the envelope was given by  $t^{-\alpha}$ , where  $\alpha \sim 0.8$ . No attempt was done in Ref. [6], however, to determine the value of  $\alpha$  accurately, or to determine its dependence or lack thereof of the parameters of the problem. The most detailed analytical study of the problem was done by Koyama and Tomimatsu in a series of three papers [7–9].

The most striking feature of the problem of tails of a massive scalar field is that the tails exist already in flat spacetime [11], because spacetime acts like a dispersive medium for a Klein-Gordon field. Based on the exact Green function, the behavior of the field in Minkowski spacetime was found in Ref. [6] to be given by

$$\psi_{\text{flat}} \sim t^{-\ell - 3/2} \sin(\mu t), \quad (1.1)$$

$\ell$  being the multipole moment of the field, and  $\mu^{-1}$  being its Compton wavelength. A derivation of Eq. (1.1) appears in the Appendix. More recently, an interesting self-similar behavior of a Klein-Gordon field in Minkowski spacetime was found in Ref. [10]. The self-similar behavior reduces to Eq. (1.1) for a field evaluated at a fixed location at very late times.

In the spacetime of a Schwarzschild black hole there are important differences. The decay rate was found in Ref. [8] to be given by

$$\psi_{\text{sch}} \sim t^{-5/6} \sin[\omega(t) \times t], \quad (1.2)$$

where the power index is independent of  $\ell$ . The time-evolving angular frequency  $\omega(t)$  approaches  $\mu$  asymptotically as  $t \rightarrow \infty$ , but is different from  $\mu$  at finite values of time. This result was generalized in Ref. [9] to any spherically symmetric, static black-hole spacetime. The details of this result have not been corroborated numerically—although qualitatively they are supported by the results of Ref. [6]—a task we undertake in this paper. In addition, we also show that this result remains correct also for a Kerr black hole, for which the power index  $-5/6$  is independent of both  $\ell$  and the spin rate of the black hole  $a/M$ . We also study the behavior of  $\omega(t)$ , and find it to be in agreement with the results of Refs. [8,9]. Similar conclusions were obtained recently also for a spinning dilaton black hole [12] (see also Ref. [13]).

## II. MASSIVE TAILS IN SCHWARZSCHILD SPACETIME

Massive fields in Schwarzschild spacetime were studied numerically (for both a linearized and for a spherical self-gravitating Klein-Gordon field) in Ref. [6] and analytically in Ref. [8]. As discussed in Ref. [8], for  $\mu M \leq 1$ , the tail regime satisfies  $t \gg \mu^{-3} M^{-2}$ ,  $M$  being the mass of the black hole, and  $t$  being the regular Schwarzschild time coordinate. Therefore, for a light scalar field, the heavier the field is, the earlier the tails are seen. (For heavy fields,  $\mu M \gg 1$ , the tail

regime is for  $t \gg M$ , independently of the mass of the field.) To facilitate numerical simulations, one is tempted then to increase the mass of the field. However, increasing the mass of the scalar field requires one also to increase the resolution, as the wavelength of the field is inversely proportional to its mass, and as one needs at least a few data points per wavelength to resolve an oscillating field. One faces then a competing effect: to decrease the computational time to an optimum, one needs to balance the scalar field mass and the resolution of the numerical code. (We do not discuss here the case of  $\mu M \gg 1$  as the required numerical resolution is beyond current practical computational limits.) One may gain insight into the above discussion, noticing, following Ref. [6], that for  $\mu \ll 1/M$ , the corresponding Compton wavelength of the scalar field  $\lambda (= \mu^{-1}) \gg M$ , that is, the field's wavelength, is much longer than the typical radius of curvature of spacetime, or the scale of inhomogeneity of curvature. We therefore expect the field at early times to evolve similarly to its evolution in flat spacetime. At later times the curvature effects become apparent, and we expect deviations from the flat-spacetime behavior. At asymptotically late times we expect to find the behavior predicted in Ref. [8]. The greater the mass term  $\mu$ , the shorter the associated Compton wavelength  $\lambda$ , and the sooner the asymptotic domain is expected. We therefore expect an oscillating field, whose envelope is a broken power law: at early times it is described by  $t^{-\ell-3/2}$  and at late times by  $t^{-5/6}$ .

The field  $\psi$  satisfies the Klein-Gordon equation

$$(\square - \mu^2) \psi = 0, \quad (2.1)$$

$\square$  being the D'Alembertian operator in curved spacetime. We define  $\psi = r\phi$  in the usual way, and decompose the field  $\psi$  into Legendre modes  $\psi = \sum \psi_\ell P_\ell(\cos \theta)$ . The radial equation that each mode  $\psi_\ell$  satisfies is then given by

$$\psi_{,uv}^\ell + \frac{1}{4} V_\ell(r) \psi^\ell = 0, \quad (2.2)$$

where the effective potential

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right) \left[ \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \mu^2 \right]. \quad (2.3)$$

To test our expectations in the context of a Schwarzschild black hole, we used a double-null code in (1+1)D, and solved the radial equation for  $\psi^\ell$ . We specified a pulse of the form

$$\psi^\ell(u) = \left[ 4 \frac{(u-u_1)(u-u_2)}{(u_2-u_1)^2} \right]^8 \quad (2.4)$$

along the ingoing segment of the characteristic hypersurface  $v=0$  for  $u_1 < u < u_2$ , and  $\psi(u)=0$  otherwise. We also took  $\psi(v)=0$  along  $u=0$ . Here,  $u$  ( $v$ ) is the usual retarded (advanced) time. The results reported on below correspond to  $u_1=20M$  and  $u_2=60M$ , but our results are unchanged also for other choices of the parameters, or for other choices of the characteristic pulse. In the following we observe the

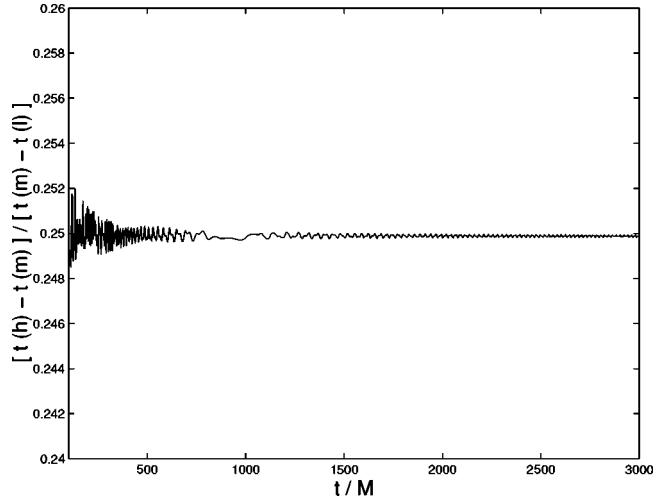


FIG. 1. Convergence test for the (1+1)D code. We find the times at which the field has zeros for three different resolutions, high  $t(h)$ , medium  $t(m)$ , and low  $t(l)$ , and compute the ratio  $[t(h) - t(m)]/[t(m) - t(l)]$  for each zero. We then plot this ratio as a function of time  $t$ . Here, we used  $N = 5, 10$ , and  $20$  grid points per  $M$  for the three resolutions. Second-order convergence corresponds to the ratio equaling 0.25.

fields along  $r_* = 0$ ,  $r_*$  being the usual Regge-Wheeler logarithmic radial (“tortoise”) coordinate.

The code is stable, and exhibits second order convergence. The convergence test is presented in Fig. 1, which shows the ratio of differences for three runs with different resolutions.

We first chose a low value of  $\mu M$ . Figure 2 shows the field along  $r_* = \text{const}$  as a function of time. The initial field here is spherical ( $\ell=0$ ), and a value of  $\mu M = 10^{-3}$  was chosen. Similar results are obtained also for other low values of  $\mu M$ . After the prompt burst and the quasinormal epoch,

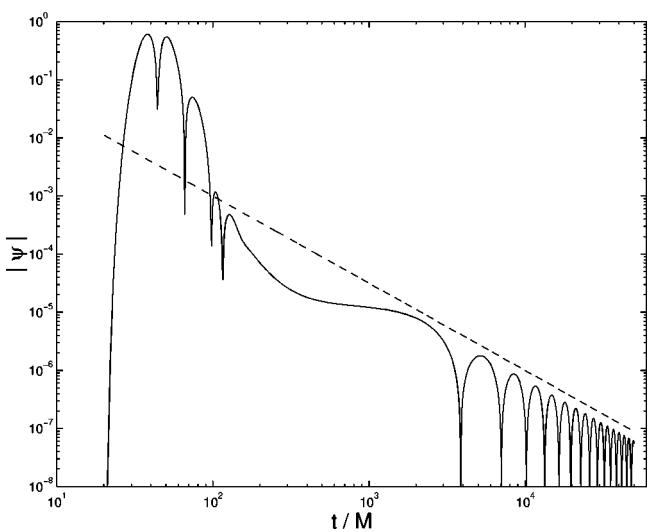


FIG. 2. Field of a massive scalar field along  $r_* = 0$ . The initial data are for  $\ell=0$ ,  $u_1=20M$  and  $u_2=60M$ , and  $N=1$ . The solid curve is the massive scalar field (with  $\mu M = 10^{-3}$ ), and the dashed line is proportional to  $(t/M)^{-3/2}$ .

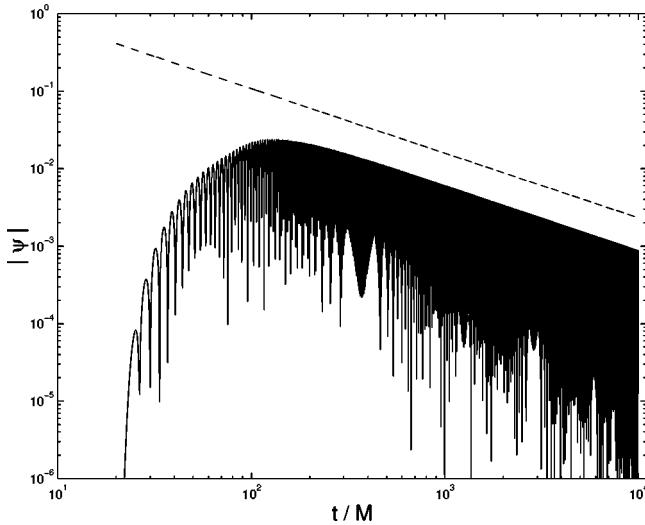


FIG. 3. Field of a massive scalar field along  $r_* = 0$ . The initial data are for  $\ell = 0$ ,  $u_1 = 20M$  and  $u_2 = 60M$ , and  $N = 4$ . The solid curve is the massive scalar field (with  $\mu M = 1$ ), and the dashed line is proportional to  $(t/M)^{-5/6}$ .

an oscillatory tail is seen. The decay rate of the envelope of the tail is very close to  $t^{-3/2}$ .

We next consider a high value of  $\mu M$ . Figure 3 shows the field for the same parameters as in Fig. 2, except that here  $\mu M = 1$ . Again, an oscillatory tail is seen, and the decay rate of the tail is very close to  $t^{-5/6}$ .

The most interesting case is that of an intermediate value of  $\mu M$ , for which both types of behavior coexist. In Fig. 4 we show the field for  $\mu M = 2.5 \times 10^{-2}$ . This figure shows an initial oscillatory decay with an envelope decaying like  $t^{-3/2}$ , which changes gradually to an oscillatory decay with a decay rate of  $t^{-5/6}$ . This is the broken power law expected: at early times the field has not noticed yet the presence of the black

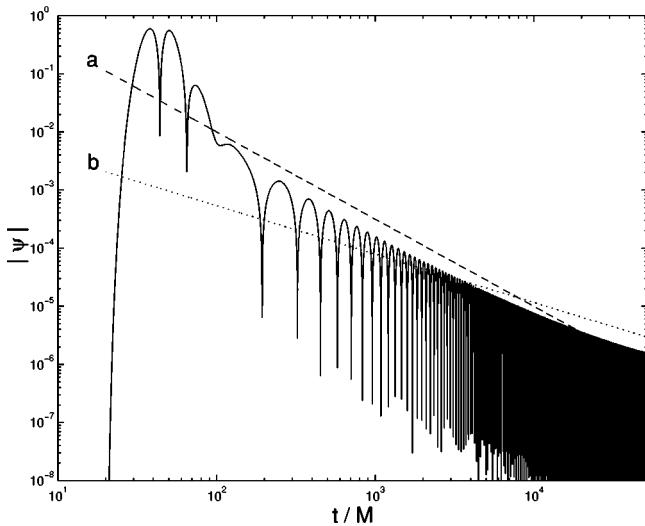


FIG. 4. Field of a massive scalar field along  $r_* = 0$ . The initial data are for  $\ell = 0$ ,  $u_1 = 20M$  and  $u_2 = 60M$ , and  $N = 4$ . The solid curve is the massive scalar field (with  $\mu M = 2.5 \times 10^{-2}$ ). The dashed line (a) is proportional to  $(t/M)^{-3/2}$ , and the dotted line (b) is proportional to  $(t/M)^{-5/6}$ .

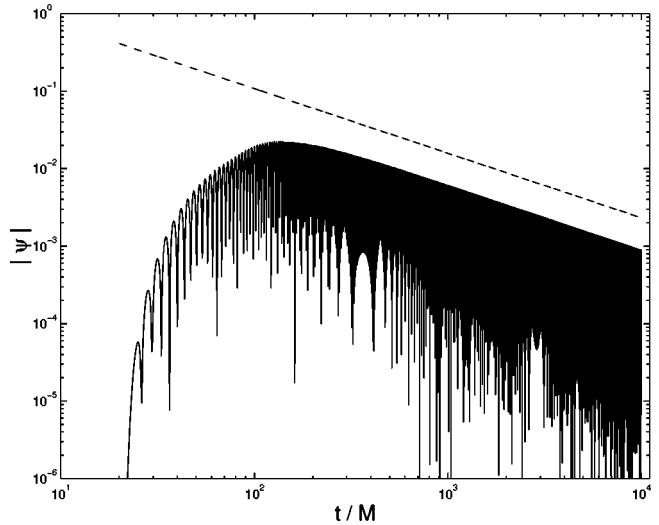


FIG. 5. Same as Fig. 3, but with  $\ell = 1$ .

hole, and therefore behaves as in flat spacetime (decay rate of  $t^{-3/2}$ ). At late times, the field takes its asymptotic decay rate of  $t^{-5/6}$ , and the onset of the asymptotic behavior can be controlled by changing the value of  $\mu M$ .

The prediction of Ref. [8] is that the asymptotic decay rate of  $t^{-5/6}$  is independent of the value of  $\ell$ . We check this in Fig. 5, which shows the same data as in Fig. 3, except that here we take  $\ell = 1$ . Again, the decay rate is  $t^{-5/6}$ , the same as with  $\ell = 0$ . We could not find any deviation from the  $t^{-5/6}$  behavior for any value of  $\ell$ . We therefore conclude that our simulations are in agreement with the prediction of Ref. [8].

In flat spacetime the period of the oscillations does not change at late times, and is given by  $T_{\text{flat}} = 2\pi/\mu$  (see the Appendix). In Schwarzschild spacetime, the period is no longer fixed. In Ref. [8], the oscillatory part of the field is given by

$$\sin[\mu t - (3/2)(2\pi\mu M)^{2/3}(\mu t)^{1/3} + \text{smaller terms}].$$

To interpret this prediction as a changing period, we rewrite the oscillatory part of the field as  $\sin[\omega(t)t]$ , where

$$\omega(t) = \mu \left[ 1 - \frac{3}{2} \left( 2\pi \frac{M}{t} \right)^{2/3} + O\left(\frac{1}{t}\right) \right]. \quad (2.5)$$

Figure 6 shows the power spectrum of the field, for the case  $\mu M = 1$ . A sharp peak at  $f = (2\pi)^{-1}$  is seen. This peak corresponds to the angular frequency equaling the mass term  $\mu$ . Notice the one-sided broadening of this peak. While no frequencies higher than  $(2\pi)^{-1}$  appear to be present, lower frequencies are. This suggests that as time progresses, the frequency increases to its asymptotic value of  $(2\pi)^{-1}$ . We test this in Fig. 7 by plotting  $1 - 2\pi/P(t)$  as a function of  $t^{-1}$ . Here,  $P(t)$  is the local period of the oscillations. We find that as  $t \rightarrow \infty$ , indeed the period decreases to its asymptotic value. Next, we consider in greater detail the rate at which the period approaches its asymptotic value. Recall that according to Ref. [8], the slope of the curve in Fig. 7 should be  $2/3$ . In Fig. 8 we display the local slope of the

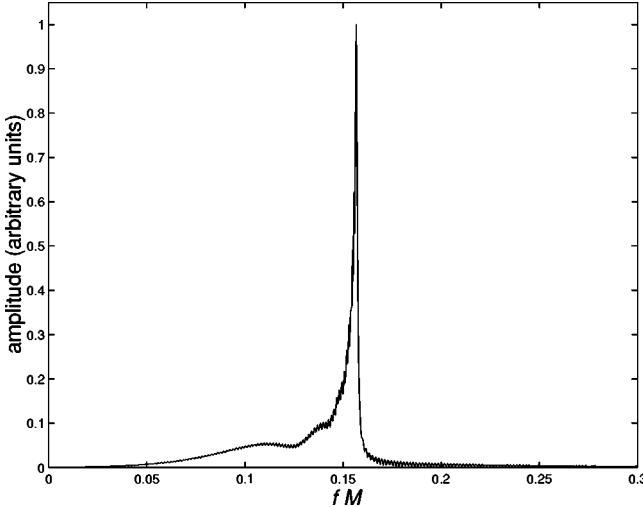


FIG. 6. Power spectrum of the field. Here,  $\mu M = 1$ , and  $N = 100$ . To obtain this plot, the field's values were first scaled by  $t^{5/6}$  to get close to a pure sinusoid, and then the Fourier transform was taken. We assume here (and below)  $M = 1$ .

curve as a function of  $t^{-1}$ . Our numerical result, of an asymptotic slope of  $0.66 \pm 0.01$ , is in agreement with the predictions of Ref. [8].

### III. MASSIVE TAILS IN KERR SPACETIME

Now we turn to the discussion of massive scalar field tails in Kerr background. We intuitively expect the late-time behavior in this case to be identical to that of the Schwarzschild case, as presented in the preceding section. The reason why the late-time tail is expected to be independent of  $a/M$  is the following: For a massless field in Kerr spacetime, there is a mixing of modes and each existing mode decays with a decay rate of  $t^{-(2\ell+3)}$ , which is the same decay rate as in the Schwarzschild case. All the modes that are not disallowed are excited, such that the overall decay rate is dominated by the existing mode with the slowest decay rate [2]. We expect

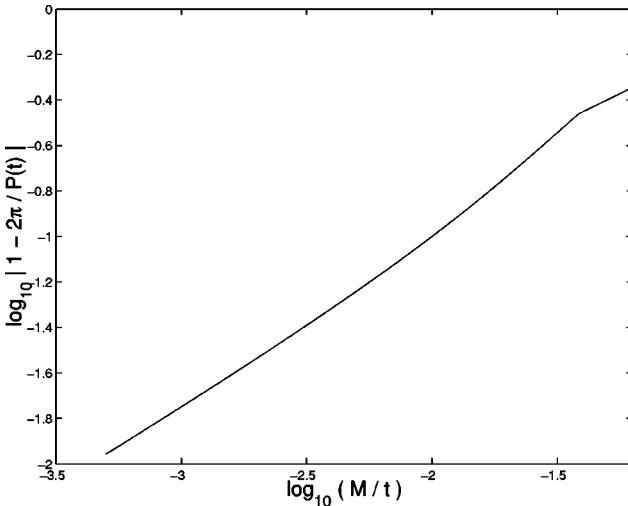


FIG. 7. The changing period,  $1 - 2\pi/P(t)$  vs  $M/t$ , for the same data as in Fig. 6.

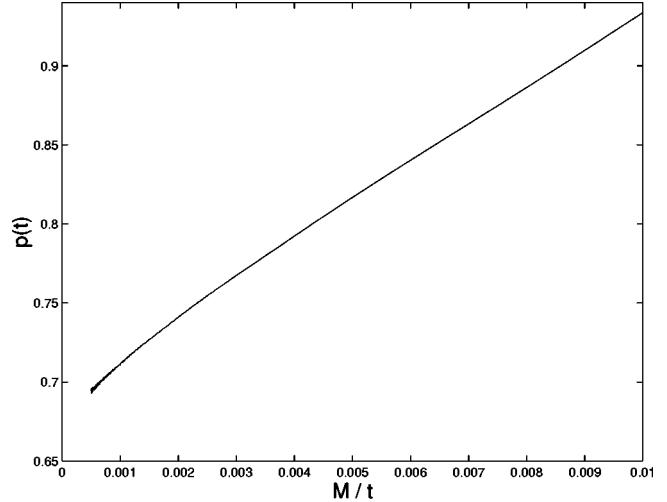


FIG. 8. The local slope of the curve in Fig. 7,  $p(t)$ , as a function of  $M/t$ .

this situation to remain basically the same also for a massive field. However, for a massive scalar field in Schwarzschild spacetime the decay rate is independent of the mode  $\ell$ . This implies that although in Kerr spacetime, mode mixing will indeed generate more modes and each mode will have the same evolution as in Schwarzschild spacetime (as in the massless case), because in the Schwarzschild case all massive field modes have the same decay rate, and they will all have the same decay rate in Kerr spacetime.

Our numerical simulations were performed using the penetrating Teukolsky code (PTC) [14], which solves the Teukolsky equation for linearized perturbations over a Kerr background in the ingoing Kerr coordinates  $(\tilde{t}, r, \theta, \tilde{\varphi})$ . These coordinates are related to the usual Boyer-Lindquist coordinates  $(t, r, \theta, \varphi)$  through  $\tilde{\varphi} = \varphi + \int a \Delta^{-1} dr$  and  $\tilde{t} = t - r + r_*$ , where  $\Delta = r^2 - 2Mr + a^2$  and  $r_* = \int (r^2 + a^2) \Delta^{-1} dr$ . The Teukolsky equation for the function  $\psi$  in the ingoing Kerr coordinates can be obtained by implementing black-hole perturbation theory (with a minor rescaling of the Kinnersley tetrad [14]). It has no singularities at the event horizon, and therefore is capable of evolving data across it. This equation is given for a massless field ( $\mu = 0$ ) by

$$\begin{aligned}
 & (\Sigma + 2Mr) \frac{\partial^2 \psi}{\partial \tilde{t}^2} - \Delta \frac{\partial^2 \psi}{\partial r^2} + 2(s-1)(r-M) \frac{\partial \psi}{\partial r} \\
 & - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \tilde{\varphi}^2} - 4Mr \frac{\partial^2 \psi}{\partial \tilde{t} \partial r} \\
 & - 2a \frac{\partial^2 \psi}{\partial r \partial \tilde{\varphi}} - i \frac{2s \cot \theta}{\sin \theta} \frac{\partial \psi}{\partial \tilde{\varphi}} + (s^2 \cot^2 \theta + s) \psi + 2[sr \\
 & + ias \cos \theta + (s-1)M] \frac{\partial \psi}{\partial \tilde{t}} = 0. \tag{3.1}
 \end{aligned}$$

For the case of interest, we set  $s=0$  in Eq. (3.1), and include a mass term  $\mu^2(r^2 + a^2 \cos^2 \theta) \psi$  on the left-hand side of Eq. (3.1). The PTC implements the numerical integration

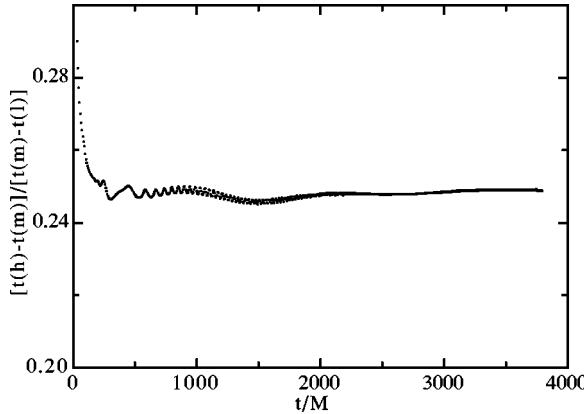


FIG. 9. Convergence test for the (2+1)D penetrating Teukolsky code. We compute the times at which the field has zeros for three separate resolutions [high  $t(h)$ , medium  $t(m)$ , and low  $t(l)$ ] and plot the ratio  $[t(h) - t(m)]/[t(m) - t(l)]$  as a function of time. The resolutions used for this test were  $M/40$ ,  $M/20$ , and  $M/10$  for the high, medium, and low resolutions, respectively. Second-order convergence is clear by the value of the convergence ratio (0.25).

of the resulting equation by decomposing it into azimuthal angular modes and evolving each such mode using a reduced (2+1)-dimensional linear partial differential equation. The results obtained from this code are independent of the choice of boundary conditions, because the inner boundary is typically placed inside the horizon, whereas the outer boundary is placed far enough away that it has no effect on the evolution. (As was shown in Ref. [15], close timelike boundaries with Sommerfeld-like boundary conditions do not allow for the evolution of late-time tails.) Typically, for the simulations performed in this study, the outer boundary was located at  $4000M$  and a grid of size  $40000 \times 32$  (radial  $\times$  angle) was used. The initial data was always chosen to be a Gaussian distribution, centered at  $50M$  and with a width of  $2M$ . The PTC is stable and exhibits second-order convergence as clear from Fig. 9.

We first demonstrate the independence of the late-time evolution of the Kerr parameter ( $a/M$ ) of the background spacetime. Figure 10 (left panels and upper panel on the right) shows tails from several different evolutions corresponding to different values of  $a/M$ , all plotted together. Each oscillatory tail shown has the expected period (about  $2\pi/\mu$ ) and a decay rate close to  $t^{-5/6}$ . The value of  $\mu M$  is chosen here to be 1.

We next demonstrate the independence of the oscillatory tail of the value of  $\ell$ . Figure 10 (right panels) shows the late-time evolution of the massive scalar field ( $\mu M = 1$ ) in a Kerr background spacetime with  $a/M = 0.6$  for several different values of  $\ell$ . There appears to be no deviation from the  $t^{-5/6}$  oscillatory tail for any value of  $\ell$ .

Last, we turn to the changing period of the oscillations in the Kerr case. In Fig. 11 we plot the power spectrum of the field for the case of  $a/M = 0.6$ ,  $\ell = 0$ , and  $\mu M = 1$ . Much like in the Schwarzschild case in the preceding section, we observe a one-sided broadening of the peak at  $(2\pi)^{-1}$ . To test whether this indicates a monotonic increase in frequency we plot in Fig. 12  $1 - 2\pi/P(t)$  as function of  $t^{-1}$ , where

$P(t)$  is the local period. Indeed, we see in the Kerr case a decreasing period, with an asymptotic rate (slope of the curve in Fig. 12 is shown in Fig. 13) of about  $0.68 \pm 0.02$ , which is consistent with the (Schwarzschild) value of  $2/3$ .

Thus, in accord with our expectations, the asymptotic late-time behavior of massive scalar fields in a Kerr spacetime is identical to that in a Schwarzschild background. Specifically, the asymptotic late-time tail is insensitive to the multipole number  $\ell$ , and also insensitive to the spin rate of the black hole. The late-time behavior of massive scalar fields in black hole spacetimes is universal. There is yet to be an analytic study in the Kerr context, akin to the one in the Schwarzschild case [7–9]. Also, it should be noted that all the numerical simulations reported on in this work were axisymmetric. In a nonaxisymmetric evolution of massive scalar fields in a Kerr spacetime, under certain conditions, a very interesting instability arises that has been studied in the frequency-domain [16]. We are hoping to return to an extensive study of that case in the time-domain elsewhere. Lastly, intriguing deviations from the results reported on here, which appear to introduce a certain low-frequency modulation at very late times, were reported on in numerical simulations [17,18]. Notably, the analytical analyses do not show any such phenomena. It remains an open question whether such phenomena are real physical ones, or a numerical artifact.

## ACKNOWLEDGMENTS

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## APPENDIX: TAILS OF A MASSIVE SCALAR FIELD IN FLAT SPACETIME

In this Appendix we derive the decay rate of Klein-Gordon tails in flat spacetime. The exact Green's function is known [11], and is given by

$$g(r,t) = \frac{\delta(t-r)}{r} - \frac{\mu J_1(\mu \sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}} \Theta(t-r), \quad (A1)$$

for a source at the origin (in space and time). Here,  $J_1$  is the Bessel function of the first kind of order 1. This Green's function can of course be integrated to find the field straightforwardly, but instead we shall use a more geometric approach to find the field.

Consider a spherical shell of radius  $r_0$ , such that the initial perturbation field is well localized in space and in time. The only requirement is that the perturbation will be regular

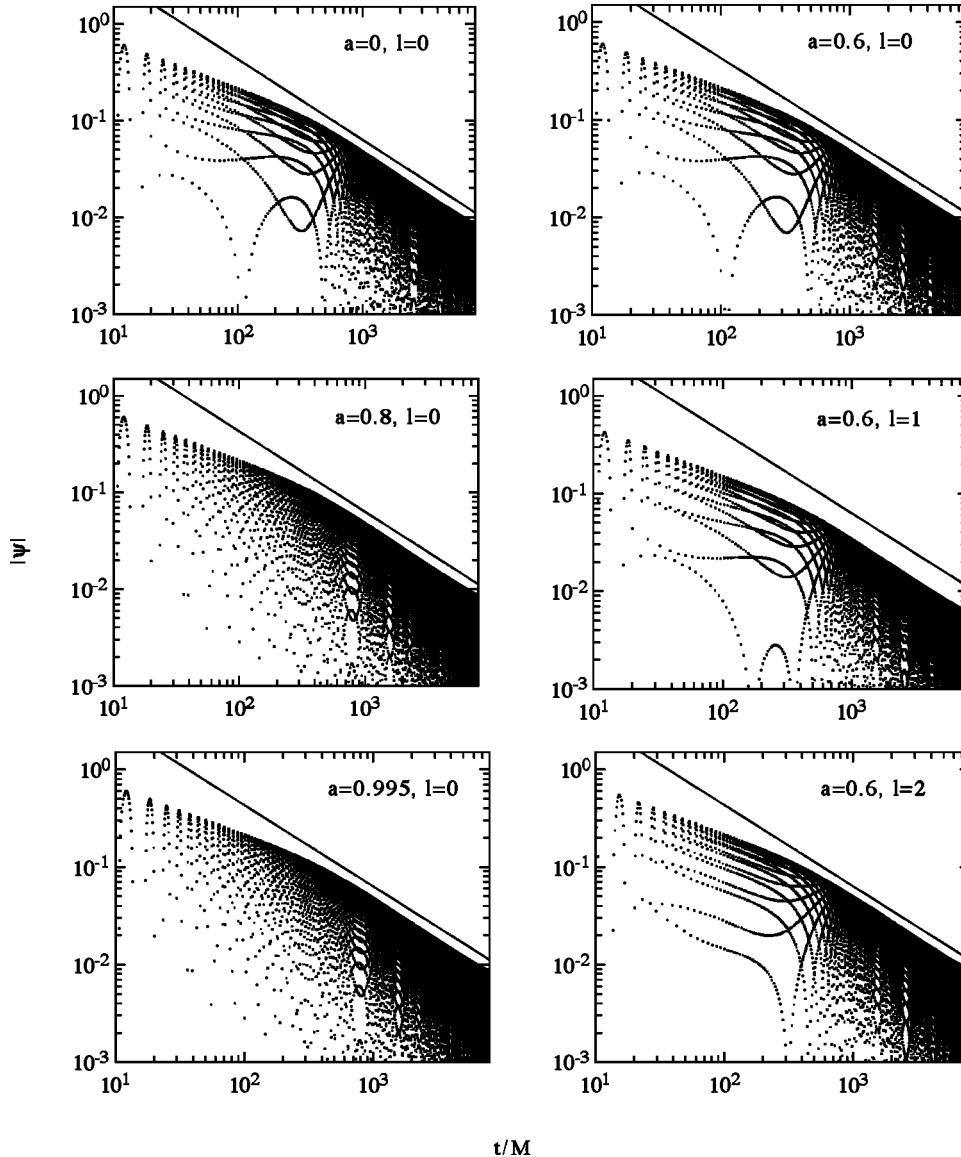


FIG. 10. Massive scalar field (with  $\mu M = 1$ ) sampled at  $r = 50M$ . The initial data have a Gaussian distribution, centered at  $50M$ , and have a width of  $2M$ . The dotted curves show the behavior of the scalar field, and the solid lines are proportional to  $(t/M)^{-5/6}$ . The sequence of plots demonstrate a clear independence of  $a/M$ , and also of the mode number  $\ell$ .

everywhere, and in particular at the location and time of the burst and at the origin  $r=0$ . The field  $\phi$  satisfies the Klein-Gordon equation

$$(\square - \mu^2) \phi = 0. \quad (\text{A2})$$

We next define, as usual,  $\psi = r\phi$ , and write the radial equation for  $\psi$ .

As the field propagates in spherical waves, the field can depend only on the spacetime interval  $s$  from the event of the burst. That is,  $\psi = \psi(s)$ . Inserting this into the radial equation, we find that  $\psi$  satisfies

$$s^2 \psi'' + s \psi' + \mu^2 s^2 \psi = 0, \quad (\text{A3})$$

a prime denoting derivative with respect to the interval  $s$ . This is just the Bessel equation, with general solution

$$\psi(s) = \tilde{c}_1 J_0(\mu s) + \tilde{c}_2 Y_0(\mu s). \quad (\text{A4})$$

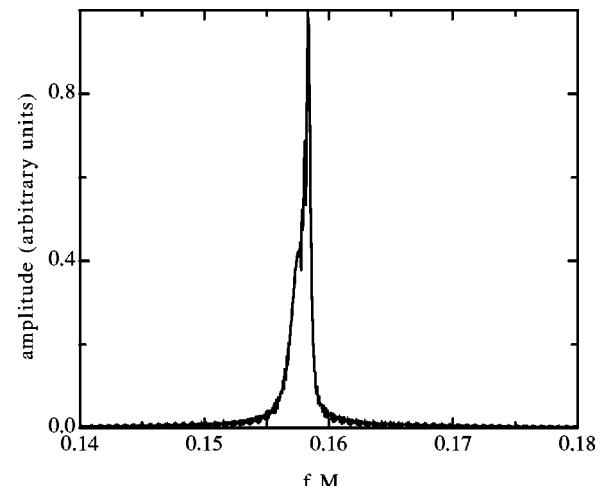


FIG. 11. Power spectrum of the field evolving in a Kerr spacetime. Here,  $\mu M = 1$ , and  $a/M = 0.6$ . To obtain this plot, the field's values were first scaled by  $t^{5/6}$  to get close to a pure sinusoid, and then the Fourier transform was taken.

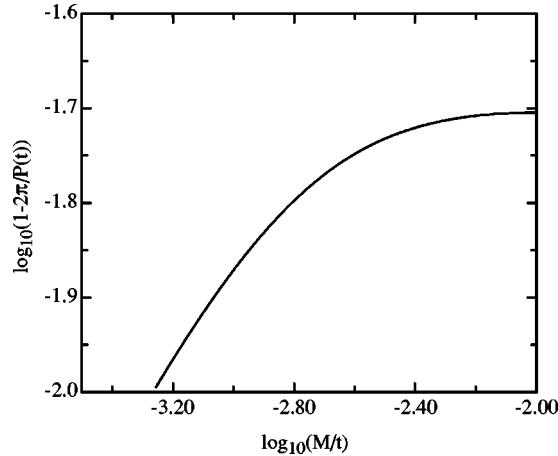


FIG. 12. The changing period in Kerr spacetime:  $1 - 2\pi/P(t)$  vs  $M/t$  for the same parameters as in Fig. 11.

Here,  $J_0$  and  $Y_0$  are the Bessel functions of the first and second kinds, respectively, of order 0. The integration constant  $\tilde{c}_2$  must vanish for the field to be regular at the event of the burst ( $s=0$ ). Therefore,

$$\psi(s) = \tilde{c}_1 J_0(\mu s). \quad (\text{A5})$$

For  $r=\text{const}$  and for  $t \gg r, r_0$ , we then find that

$$\psi \approx c_1 \frac{\cos(\mu t - \pi/4)}{(\mu t)^{1/2}}. \quad (\text{A6})$$

This solution does not satisfy the regularity requirement, as the field  $\phi$  diverges at the spatial origin  $r=0$ . However, the field  $\phi$  is required to be regular at  $r=0$ , especially at late times ( $\mu t \gg 1$ ).

To obtain the physical solution we superimpose solutions to the radial equation, such that the field is regular at the origin. This can be achieved by an “image charge.” The superposition

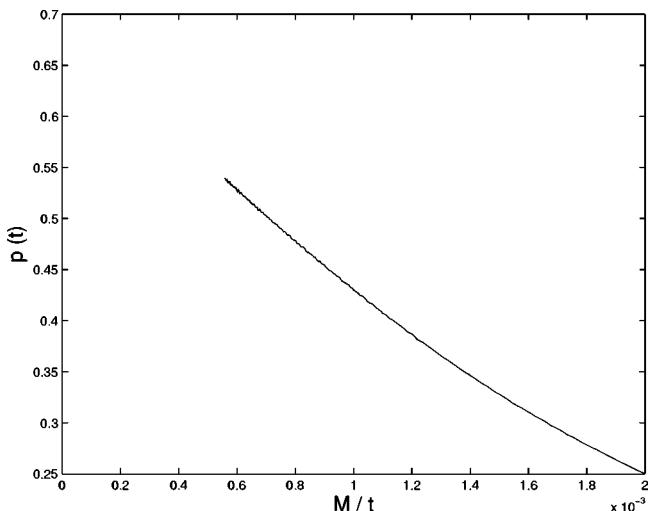


FIG. 13. The local slope of the curve in Fig. 12,  $p(t)$ , as a function of  $M/t$ .

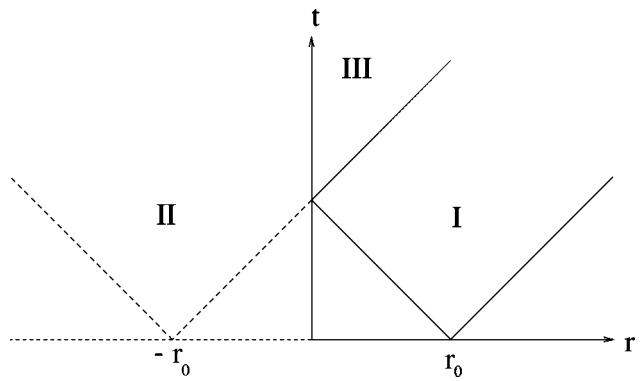


FIG. 14. The construction of the “image charge” solution. The burst of the spherical wave is at the event  $(0, r_0)$ . The domain of dependence of this event is the union of regions I and III in the diagram. As this solution is not regular along  $r=0$  we introduce an “image charge,” i.e., a burst with the opposite sign at the fictitious event  $(0, -r_0)$ . The field due to this image solution in region II is, of course, unphysical, but in region III it is, and is superimposed with the solution of the original burst in that region.

$$\psi(s) = \tilde{c}_1 [J_0(\mu s_1) - J_0(\mu s_2)] \quad (\text{A7})$$

is the requested solution, where  $s_1^2 = t^2 - (r - r_0)^2$  and  $s_2^2 = t^2 - (r + r_0)^2$ . This corresponds to an “image charge” at  $r = -r_0$ . Obviously,  $r = -r_0$  is not a physical point in spacetime. It is a mathematical way to write a formal solution, where in the physical part of the domain of dependence of the superimposed bursts the solution is the one sought (see Fig. 14). In fact, this is just a mathematical way to specify regular boundary conditions along  $r=0$ .

For  $r=\text{const}$  and for  $t \gg r, r_0$ , we then find that

$$\psi(t \gg r, r_0) \approx c_1 \frac{\cos(\mu t - \pi/4)}{(\mu t)^{3/2}} r r_0. \quad (\text{A8})$$

The phase is unimportant, such that we finally get the result

$$\phi(t \gg r, r_0) \approx c_1 r_0 \frac{\sin(\mu t)}{(\mu t)^{3/2}}. \quad (\text{A9})$$

The fall off rate of  $t^{-3/2}$  is the one reported on in Ref. [11].

To find the time dependence of higher  $\ell$  modes, we find the derivative of the spherical solution with respect to the Cartesian coordinate  $z$ . Recall that on the equatorial plane,  $\partial P_\ell(\cos \theta)/\partial z = -(\ell+1)P_{\ell+1}(\cos \theta)/r$ . We can thus generate  $\psi_1$  from  $\psi_0$  by calculating  $\partial \psi_{\ell=0}/\partial z$ , and by the uniqueness of the solution that should give us the dipole solution  $\psi_{\ell=1}$ . Specifically, notice that

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z} \frac{\partial s}{\partial r} \frac{\partial}{\partial s} = -\frac{r}{s} \cos \theta \frac{\partial}{\partial s}.$$

Notice the factor  $1/s$  in this expression. This implies that whenever a derivative with respect to  $z$  is taken, the exponent of  $t$  in the denominator increases by 1. (This is the case because of the sinusoidal function in the numerator. The leading term in  $t^{-1}$  comes from its differentiation, and then

the exponent of the denominator does not change. It is increased by 1 because of this  $1/s$  factor.)

Carrying  $\ell$  times this differentiation with respect to  $z$ , we find that

$$\psi_\ell(t \gg r, r_0) \sim \frac{\sin(\mu t)}{t^{\ell+3/2}}. \quad (\text{A10})$$

The reason why the Huygens principle fails in this case is that the phase velocity of a plane wave satisfying the Klein-Gordon equation is

$$v_{\text{ph}} = \frac{1}{\sqrt{1 - (\mu/\omega)^2}}, \quad (\text{A11})$$

so that whenever  $\mu \neq 0$ , different frequencies which make up a wave packet travel at different speeds. As noted in Ref. [11], it is not surprising that the associated plane waves do not arrive at the observation point with the same relative phase they had at the beginning. In fact, spacetime behaves like a dispersive medium for the Klein-Gordon equation, as the wave number  $k = \omega \sqrt{1 - (\mu/\omega)^2}$  is no longer a linear function of  $\omega$ .

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